

Kählerian Reduction in Steps

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Abstract

We study Hamiltonian actions of compact Lie groups K on Kähler manifolds which extend to a holomorphic action of the complexified group $K^{\mathbb{C}}$. For a closed normal subgroup L of K we show that the Kählerian reduction with respect to L is a stratified Hamiltonian Kähler $K^{\mathbb{C}}/L^{\mathbb{C}}$ -space whose Kählerian reduction with respect to K/L is naturally isomorphic to the Kählerian reduction of the original manifold with respect to K .

1 Introduction

Reduction of variables for physical systems with symmetries is a fundamental concept in classical Hamiltonian dynamics. It is based on Noether's principle that every 1-parameter group of symmetries of a physical system corresponds to a constant of motion. The mathematical formalisation is known as the Marsden-Weinstein-Reduction or *symplectic reduction*. Consider a symplectic manifold X with a smooth symplectic form ω and a smooth action of a Lie group K on X by ω -isometries. Assume that there exists a smooth map μ from X into the dual space \mathfrak{k}^* of the Lie algebra \mathfrak{k} of K such that

- a) for every $\xi \in \mathfrak{k}$, the function $\mu^\xi : X \rightarrow \mathbb{R}$, $\mu^\xi(x) = \mu(x)(\xi)$, fulfills $d\mu^\xi = \iota_{\xi_X} \omega$, where ξ_X denotes the vector field on X induced by the action of K on X and ι_{ξ_X} denotes contraction with respect to ξ_X ,
- b) the map $\mu : X \rightarrow \mathfrak{k}^*$ is equivariant with respect to the action of K on X and the coadjoint representation Ad^* of K on \mathfrak{k}^* , i.e. for all $x \in X$ and for all $k \in K$ we have $\mu(k \cdot x) = \text{Ad}^*(k)(\mu(x))$.

The map $\mu : X \rightarrow \mathfrak{k}^*$ is called a (K -equivariant) *momentum map* and the action of K on X is called *Hamiltonian*.

The motion of a classical particle is given by a Hamiltonian function H . More precisely, let H be a smooth function on X . The time evolution of the underlying physical system is described

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by the flow of the vector field V_H associated to the Hamiltonian H via the equation $dH = \iota_{V_H}\omega$. A Hamiltonian system with symmetries is a system (X, ω, K, μ, H) where H is invariant with respect to the K -action. In this case it follows from a) that

$$d\mu^\xi(V_H) = \omega(\xi_X, V_H) = -dH(\xi_X) = 0$$

holds for all $\xi \in \mathfrak{k}$. This implies Noether's principle in the following geometric formulation: every component μ^ξ of the momentum map is a constant of motion for the system described by H .

The previous considerations imply that level sets of μ are invariant under the flow of V_H . In many cases questions can be organised so that $\mu^{-1}(0)$ is the momentum fibre of interest. By b), the group K acts on the level set $\mu^{-1}(0)$. Let us first consider this action on an infinitesimal level. If we fix $x_0 \in \mathcal{M} := \mu^{-1}(0)$, it follows from a) that $\ker(d\mu(x_0)) = (\mathfrak{k} \bullet x_0)^\perp$, where $\mathfrak{k} \bullet x_0 = \{\xi_X(x_0) \mid \xi \in \mathfrak{k}\}$ and $^\perp$ denotes perpendicular with respect to ω . The optimal situation appears if μ has maximal rank at x_0 . In this case, \mathcal{M} is smooth at x_0 and $T_{x_0}\mathcal{M} = \ker(d\mu(x_0)) = (\mathfrak{k} \bullet x_0)^\perp$ holds. It follows that \mathcal{M} is a coisotropic K -stable submanifold of X and the symplectic form ω_{x_0} on $T_{x_0}\mathcal{M}$ induces a symplectic form $\tilde{\omega}_{x_0}$ on $T_{x_0}\mathcal{M}/T_{x_0}\mathcal{M}^\perp = T_{x_0}\mathcal{M}/\mathfrak{k} \bullet x_0$. These observations imply that once the space \mathcal{M}/K is smooth, it will be a symplectic manifold. This is the content of the *Marsden-Weinstein-Theorem* (see [MW74]):

If K acts freely and properly on \mathcal{M} , the quotient \mathcal{M}/K is a symplectic manifold whose symplectic form $\tilde{\omega}$ is characterised via the equation

$$\pi^*\tilde{\omega} = i_{\mathcal{M}}^*\omega.$$

Here, $\pi : \mathcal{M} \rightarrow \mathcal{M}/K$ denotes the quotient map and $i_{\mathcal{M}} : \mathcal{M} \rightarrow X$ is the inclusion. Furthermore, the restriction of the K -invariant Hamiltonian H to \mathcal{M} induces a smooth function \tilde{H} on \mathcal{M}/K . The Hamiltonian system on $(\mathcal{M}/K, \tilde{\omega})$ associated to \tilde{H} captures the essential (symmetry-independent) properties of the original K -invariant system that was given by H .

The Marsden-Weinstein construction is natural in the sense that it can be done in steps. This means that for a normal closed subgroup L of K , the restricted momentum map $\mu_L : X \rightarrow \mathfrak{l}^*$ is K -equivariant, the induced K -action on $\mu_L^{-1}(0)/L$ is Hamiltonian with momentum map $\tilde{\mu}$ induced by μ and the symplectic reduction $\tilde{\mu}^{-1}(0)/K$ is symplectomorphic to \mathcal{M}/K .

Removing the restrictive regularity assumptions of [MW74], it is proven in [SL91] that symplectic reduction can be carried out for general group actions of compact Lie groups yielding stratified symplectic quotient spaces \mathcal{M}/K , i.e. stratified spaces where all strata are symplectic manifolds. The paper [HHL94] proposed an approach to this singular symplectic reduction based on embedding symplectic manifolds into Kähler manifolds and a Kähler reduction theory for Kähler manifolds. Roughly speaking, \mathcal{M}/K is realised as a locally semialgebraic subspace of a Kähler space Q and the symplectic structure on \mathcal{M}/K is given by restriction. Inspecting the proofs one sees that the construction of [HHL94] is compatible with reduction in steps.

Our interest here is to study the problem of reduction in steps in a Kählerian context using techniques related to the complex geometry and invariant theory for the complexification $K^\mathbb{C}$ of a compact Lie group K .

Holomorphic actions of compact groups K on complex spaces X very often extend to holomorphic actions of the complexification $K^{\mathbb{C}}$, at least in the sense that there exists an open K -equivariant embedding $X \hookrightarrow X^c$ of X into a holomorphic $K^{\mathbb{C}}$ -space X^c (see [HI97]). In this note, we consider the case where $K^{\mathbb{C}}$ already acts on X . More precisely, we consider Kähler manifolds X with a Hamiltonian action of a compact Lie group K that extends to a holomorphic action of the complexification $K^{\mathbb{C}}$. In this setup, an extensive quotient theory has been developed. See Section 1.1 for a survey of the results used in this note. Due to the action of the complex group $K^{\mathbb{C}}$, it is possible to introduce a notion of semistability. The reduced space \mathcal{M}/K carries a complex structure given by holomorphic invariant theory for the action of $K^{\mathbb{C}}$ on the set of semistable points. The reduced symplectic structure is compatible with this complex structure, i.e. \mathcal{M}/K is a Kähler space. Since the quotient \mathcal{M}/K in general will be singular, the Kähler structure is locally given by continuous strictly plurisubharmonic functions that are smooth along the natural stratification of \mathcal{M}/K given by the orbits of the action of K on \mathcal{M} . We emphasize at this point that both the complex structure and the Kähler structure are defined globally, i.e. across the strata. We call \mathcal{M}/K the *Kählerian reduction* of X by K . This reduction theory works more generally for stratified Kählerian complex spaces, however, in order to reduce technical difficulties and not to obscure the main principles at work, we will restrict our attention to actions on complex manifolds. Nevertheless, we also have to consider complex spaces which appear as quotients by normal subgroups of $K^{\mathbb{C}}$.

Using the approach of [HHL94] and the quotient theory for complex-reductive group actions, we show that Kählerian reduction can be done in steps. More precisely, we prove that the Kählerian reduction of X by a normal subgroup L of K is a stratified Hamiltonian Kähler $K^{\mathbb{C}}$ -space and that its Kählerian reduction is isomorphic to the reduction of X by K (see Theorem 2.1). This shows that reduction in steps respects the Kähler geometry. It also exhibits the class of Kählerian stratified spaces as natural for a Kählerian reduction theory. Furthermore, we look carefully at the stratifications obtained on the various quotient spaces.

1.1 Reductive group actions on Kähler spaces

As we have noted above, symplectic reduction of Kähler manifolds yields spaces that are endowed with a complex structure. This is due to a close relation between the quotient theory of a compact Lie group K and the quotient theory for its complexification $K^{\mathbb{C}}$ which we will now explain.

In this paper a complex space refers to a reduced complex space with countable topology. If G is a Lie group, then a *complex G -space* Z is a complex space with a real-analytic action $G \times Z \rightarrow Z$ which for fixed $g \in G$ is holomorphic. For a complex Lie group G a *holomorphic G -space* Z is a complex G -space such that the action $G \times Z \rightarrow Z$ is holomorphic.

We note that given a compact real Lie group K , there exists a complex Lie group $K^{\mathbb{C}}$ containing K as a closed subgroup with the following universal property: given a Lie homomorphism $\varphi : K \rightarrow H$ into a complex Lie group H , there exists a holomorphic Lie homomorphism $\varphi^{\mathbb{C}} : K^{\mathbb{C}} \rightarrow H$ extending φ .

If not only K but also $K^{\mathbb{C}}$ acts holomorphically on a manifold X , it is natural to relate the

quotient theory of K to the quotient theory of $K^{\mathbb{C}}$. Due to the existence of non-closed $K^{\mathbb{C}}$ -orbits, in contrast to actions of K , actions of $K^{\mathbb{C}}$ will in general not give rise to reasonable orbit spaces. However, as we will see, it is often possible to find an open subset U of X and a complex space Y that parametrises closed $K^{\mathbb{C}}$ -orbits in U . More precisely, we call a complex space Y an *analytic Hilbert quotient* of U by the action of $K^{\mathbb{C}}$ if there exists a $K^{\mathbb{C}}$ -invariant Stein holomorphic map $\pi : U \rightarrow Y$ with $(\pi_* \mathcal{O}_U)^{K^{\mathbb{C}}} = \mathcal{O}_Y$. Here, a *Stein map* is a map such that inverse images of Stein subsets are again Stein. It can be shown (see [HL94], for example) that Y is the quotient of U by the equivalence relation

$$x \sim y \quad \text{if and only if} \quad \overline{K^{\mathbb{C}} \cdot x} \cap \overline{K^{\mathbb{C}} \cdot y} \neq \emptyset.$$

Analytic Hilbert quotients are universal with respect to $K^{\mathbb{C}}$ -invariant holomorphic maps, i.e. given a $K^{\mathbb{C}}$ -invariant holomorphic map $\varphi : U \rightarrow Z$ into a complex space Z , there exists a uniquely determined holomorphic map $\bar{\varphi} : Y \rightarrow Z$ such that $\varphi = \bar{\varphi} \circ \pi$. It follows that an analytic Hilbert quotient of U by $K^{\mathbb{C}}$ is unique up to biholomorphism once it exists. We denote it by $U // K^{\mathbb{C}}$.

The theory of analytic Hilbert quotients is interwoven with the Kählerian quotient theory as follows (see [HL94]): let K be a compact Lie group with Lie algebra \mathfrak{k} , X a holomorphic $K^{\mathbb{C}}$ -manifold. Assume that the action of K is Hamiltonian with respect to a K -invariant Kähler form on X with K -equivariant momentum map $\mu : X \rightarrow \mathfrak{k}^*$. In this situation, we call X a *Hamiltonian Kähler $K^{\mathbb{C}}$ -manifold*. Let $\mathcal{M} = \mu^{-1}(0)$ and define

$$X(\mathcal{M}) := \{x \in X \mid \overline{K^{\mathbb{C}} \cdot x} \cap \mathcal{M} \neq \emptyset\}.$$

A point $x \in X(\mathcal{M})$ is called *semistable*. The set $X(\mathcal{M})$ is open in X and an analytic Hilbert quotient Q of $X(\mathcal{M})$ by $K^{\mathbb{C}}$ exists. Each fibre of the quotient map $\pi : X(\mathcal{M}) \rightarrow Q$ contains a unique closed $K^{\mathbb{C}}$ -orbit which is the unique orbit of minimal dimension in that fibre. This closed $K^{\mathbb{C}}$ -orbit intersects \mathcal{M} in a unique K -orbit, $K \cdot x$. The isotropy group $(K^{\mathbb{C}})_x$ of points $x \in \mathcal{M}$ is complex-reductive and equal to the complexification of K_x . The analytic Hilbert quotient $X(\mathcal{M}) // K^{\mathbb{C}}$ is related to the Kählerian reduction \mathcal{M}/K by the following fundamental commutative diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathbb{C}} & \xrightarrow{\iota} & X(\mathcal{M}) \\ \pi_K \downarrow & & \downarrow \pi \\ \mathcal{M}/K & \xrightarrow[\simeq]{\bar{\iota}} & Q. \end{array} \tag{1}$$

Here, the induced map $\bar{\iota}$ is a homeomorphism. Hence, the symplectic reduction \mathcal{M}/K has a complex structure induced via the homeomorphism $\bar{\iota}$. The inverse of $\bar{\iota}$ is induced via a retraction $\psi : X(\mathcal{M}) \rightarrow \mathcal{M}$ that is related to the stratification of X via the gradient flow of the norm square of the momentum map (see [Nee85], [Kir84], and [Sch89]).

Using this two-sided picture of the quotient it can be shown using the techniques of [HHL94] that \mathcal{M}/K carries a natural Kählerian structure that is smooth along the *orbit type stratification*.

1.2 Stratifying holomorphic G -spaces

Invariant stratifications are a powerful tool in the study of group actions and their quotient spaces (see [Lun73] and [Sch80]). We will recall the definitions and basic properties of these stratifications.

Definition 1.1. A *complex stratification* of a complex space X is a countable, locally finite covering of X by disjoint subspaces (the so called *strata*) $\mathbb{S} = (S_\gamma)_{\gamma \in \Gamma}$ with the following properties

1. each stratum S_γ is a locally closed submanifold of X that is Zariski-open in its closure,
2. the boundary $\partial S_\gamma = \bar{S}_\gamma \setminus S_\gamma$ of each stratum S_γ is a union of strata of lower dimension.

Example 1.2. The singular set X_{sing} of a complex space X is a closed complex subspace of smaller dimension. Iterating this procedure, i.e. considering the singular set of X_{sing} , we obtain a natural stratification on X . If a Lie group G acts holomorphically on X , this stratification is G -invariant.

We now consider stratifications related to group actions. Let G be a complex-reductive Lie group. Let X be a holomorphic G -manifold such that the analytic Hilbert quotient $\pi_G : X \rightarrow X//G$ exists. Let p be a point in $X//G$. The fibre $\pi_G^{-1}(p)$ over p contains a unique closed orbit. We denote this orbit by $C(p)$. We say that p is of *G -orbit-type* (G_1) if the stabilizer of one (and hence any point) in $C(p)$ is conjugate to G_1 in G .

The next result follows from the holomorphic slice theorem (see [Hei91]):

Proposition and Definition 1.3. *Let X be a holomorphic G -manifold such that the analytic Hilbert quotient $\pi_G : X \rightarrow X//G$ exists. Then, each connected component of the set of points of orbit type (G_1) is a locally closed manifold and the corresponding decomposition of $X//G$ is a stratification of X which we call the orbit type stratification of $X//G$.*

We obtain a related G -invariant complex stratification of X by stratifying the preimage $\pi_G^{-1}(S_\gamma)$ of each orbit type stratum $S_\gamma \subset X//G$ as a complex space.

Over a stratum S_γ , the structure of X and of the quotient $\pi_G : X \rightarrow X//G$ is particularly simple. More precisely, let S_γ be a stratum of $X//G$ and let (G_0) be the conjugacy class of isotropy groups corresponding to S_γ . Let us assume for the moment that $X//G = S_\gamma$. Then, the slice theorem implies that each point $q \in X//G$ has an open neighbourhood U such that $\pi_G^{-1}(U)$ is G -equivariantly biholomorphic to $G \times_{G_0} Z$, where Z is a locally closed G_0 -stable Stein submanifold of X . The union of the closed orbits in $\pi_G^{-1}(U)$ is equal to $G \times_{G_0} Z^{G_0} \cong G/G_0 \times Z^{G_0}$. Noticing that $U \cong (G \times_{G_0} Z)//G \cong Z//G_0 \cong Z^{G_0}$, we see that the set of closed orbits in $\pi_G^{-1}(S_\gamma)$ is a smooth fibre bundle over S_γ with typical fibre G/G_0 and structure group G .

If $X//G$ is irreducible, there exists a Zariski-open and dense stratum S_{princ} in $X//G$, called the *principal stratum*. It corresponds to the minimal conjugacy class (G_0) of isotropy groups of closed orbits, i.e. if x is any point X with closed G -orbit, G_0 is conjugate in G to a subgroup of G_x .

In the Hamiltonian setup there is a stratification related to the action of K on the momentum zero fibre: let X be a Hamiltonian Kähler $K^\mathbb{C}$ -space. Assume that $X = X(\mathcal{M})$. Decompose the quotient \mathcal{M}/K by orbit types for the action of K on \mathcal{M} . It has been shown in a purely symplectic setup in [SL91] that this defines a stratification of \mathcal{M}/K . Since \mathcal{M}/K is homeomorphic to $X//K^\mathbb{C}$ (see Diagram (1)) this defines a second decomposition of $X//K^\mathbb{C}$. However, it can be shown that the two constructions yield the same result: given a stratum S corresponding to K -orbit type (K_1) , it coincides with one of the connected components of the set of points in $X//K^\mathbb{C}$ with orbit type $(K_1^\mathbb{C})$ (see [Sja95]).

1.3 Kähler reduction

In this section we recall the basic definitions necessary for Kählerian reduction theory. Due to the presence of singularities in the spaces that we will consider, Kähler structures are defined in terms of strictly plurisubharmonic functions.

Definition 1.4. Let Z be a complex space. A continuous function $\rho : Z \rightarrow \mathbb{R}$ is called *plurisubharmonic*, if for every holomorphic map φ from the unit disc D in \mathbb{C} to Z , the pullback $\varphi^*(\rho)$ is subharmonic on D , i. e. for each $0 < r < 1$ the *mean value inequality*

$$\varphi^*(\rho)(0) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi^*(\rho)(re^{i\theta}) d\theta$$

holds.

A *perturbation* of a continuous function $\rho : Z \rightarrow \mathbb{R}$ at a point $x \in Z$ is a function $\rho + f$, where f is smooth and defined in some neighbourhood U of x . The function ρ is said to be *strictly plurisubharmonic* if for every perturbation $\rho + f$ there exist $\varepsilon > 0$ and a perhaps smaller neighbourhood V of x such that $\rho + \varepsilon f$ is plurisubharmonic on V .

Remark 1.5. If Z is a complex manifold and $\rho : Z \rightarrow \mathbb{R}$ is smooth, then ρ is strictly plurisubharmonic if and only if its Levi form $\frac{i}{2}\partial\bar{\partial}\rho$ is positive definite, i.e. if it defines a Kähler form.

Definition 1.6. A *Kähler structure* on a complex space Z is given by an open cover (U_j) of Z and a family of strictly plurisubharmonic functions $\rho_j : U_j \rightarrow \mathbb{R}$ such that the differences $\rho_j - \rho_k$ are *pluriharmonic* on $U_{jk} := U_j \cap U_k$ in the sense that there exists a holomorphic function $f_{jk} \in \mathcal{O}(U_{jk})$ with $\rho_j - \rho_k = \operatorname{Re}(f_{jk})$. Two Kähler structures (U_j, ρ_j) and $(\tilde{U}_k, \tilde{\rho}_k)$ are considered equal if there exists a common refinement (V_l) of (U_j) and (\tilde{U}_k) such that $(\rho_l - \tilde{\rho}_l)|_{V_l}$ is pluriharmonic for every l .

Again, we remark that the definition made above coincides with the usual definition of a Kähler form on a complex manifold if all the ρ_j 's are assumed to be smooth. For more information on strictly plurisubharmonic functions we refer to [Var89] and [FN80].

We have seen that analytic Hilbert quotients of manifolds have a natural stratification by orbit types. Taking into account this additional structure, we make the following definition.

Definition 1.7. A complex space Z is called a *stratified Kähler space* if there exists a complex stratification $\mathbb{S} = (S_\gamma)_{\gamma \in \Gamma}$ on Z which is finer than the stratification of Z as a complex space and there exists a Kähler structure $\omega = (U_\alpha, \rho_\alpha)_{\alpha \in I}$ on Z such that $\rho_\alpha|_{S_\gamma \cap U_\alpha}$ is smooth.

The following theorem is a special case of results proven in [HHL94] and [HL94]:

Theorem 1.8. *Let K be a compact Lie group and $G = K^\mathbb{C}$ its complexification. Let X be a Hamiltonian Kähler $K^\mathbb{C}$ -manifold with $X = X(\mathcal{M})$. We denote the quotient map by $\pi_G : X \rightarrow X//G$. Let $\mathbb{S}_G^{X//G}$ be the orbit type stratification of $X//G$ defined above. Then there exists a Kähler structure $\tilde{\omega} = (U_\alpha, \tilde{\rho}_\alpha)_{\alpha \in I}$ on $X//G$ with the following properties:*

1. *the triple $(X, \mathbb{S}_G^{X//G}, \tilde{\omega})$ is a stratified Kähler space,*
2. *there exist smooth functions $\rho_\alpha : \pi_H^{-1}(U_\alpha) \rightarrow \mathbb{R}$ with $\omega|_{\pi_H^{-1}(U_\alpha)} = 2i\partial\bar{\partial}(\rho_\alpha)$ such that the following equality holds: $\pi_G^*(\tilde{\rho}_\alpha)|_{\mathcal{M} \cap \pi_G^{-1}(U_\alpha)} = \rho_\alpha|_{\mathcal{M} \cap \pi_G^{-1}(U_\alpha)}$.*

We recall the construction of the reduced Kähler structure. With the notation of the previous theorem, the key technical result for the construction can be stated as follows:

Proposition 1.9. *Let $x \in \mathcal{M}$. Then, there exists a π_G -saturated neighbourhood U of x in X , such that the Kähler structure ω is given by a smooth strictly plurisubharmonic function $\rho : U \rightarrow \mathbb{R}$. The function ρ is an exhaustion along every fibre of π_G that is contained in U . Furthermore, the restriction of the momentum map μ to U fulfills*

$$\mu^\xi(x) = \mu_\rho^\xi(x) := \left. \frac{d}{dt} \right|_{t=0} \rho(\exp(it\xi) \cdot x) \quad \forall \xi \in \mathfrak{k}, \forall x \in U. \quad (2)$$

The set $\mathcal{M} \cap U$ coincides with the set of critical points for the restriction of ρ to fibres of the quotient map π_G .

Remark 1.10. If a momentum map μ fulfills equation (2) for some function ρ , we say that μ is *associated to ρ* and we write $\mu = \mu_\rho$.

The Kähler structure on the quotient $X(\mathcal{M})//G$ is constructed as follows: every point $y \in X//G$ has a neighbourhood of the form $U//G$ which has the properties of Proposition 1.9. The restriction of ρ to $\mathcal{M} \cap U$ induces a continuous function $\tilde{\rho}$ on $U//G$. We will see that it is strictly plurisubharmonic and smooth along the strata of $\mathbb{S}_G^{X//G}$. Let S be a stratum. Let Y be the set of closed G -orbits in $\pi_G^{-1}(S)$. This is smooth and contains $\mathcal{M} \cap \pi_G^{-1}(S)$ as a smooth submanifold. We know that ω and hence ρ are smooth along Y . The quotient map $\pi_K : \mathcal{M} \cap \pi_G^{-1}(S) \rightarrow S$ is a smooth submersion, hence $\tilde{\rho}$ is smooth along each stratum S . Over S , we can assume that $X = Y = G/\Sigma \times X//G$. For $z_0 \in \mathcal{M}$, we have

$$T_{z_0}(\mathcal{M}) = T_{z_0}(K \cdot z_0) \oplus T_{z_0}(K^\mathbb{C} \cdot z_0)^\perp,$$

where \perp denotes the perpendicular with respect to the Riemannian metric associated to ω . This allows us to construct a smooth section $\sigma : X//G \rightarrow G/\Sigma \times X//G$ for π with image in \mathcal{M} in such

a way that the differential of σ at $\pi(z_0)$ is a complex linear isomorphism from $T_{\pi(z_0)}(X//G)$ to $T_{z_0}(K^\mathbb{C} \bullet z_0)^\perp$. Let $\sigma(w) = (\eta(w), w)$. Let \tilde{f} be a smooth function near $\pi(z_0)$. Since ρ is strictly plurisubharmonic there exists an $\epsilon > 0$ such that $\rho_\epsilon := \rho + \epsilon(\tilde{f} \circ \pi)$ is plurisubharmonic on X . By construction, $\tilde{\rho}_\epsilon := \tilde{\rho} + \epsilon\tilde{f}$ is equal to $\rho_\epsilon \circ \sigma$. Now use the chain rule, the fact that $\rho_\epsilon|_{K^\mathbb{C} \bullet z}$ is critical at $K^\mathbb{C} \bullet z \cap \mathcal{M}$ and $d\eta(\pi(z_0)) = 0$ to show that

$$\frac{\partial^2 \tilde{\rho}_\epsilon}{\partial w_i \partial \bar{w}_j}(\pi(z_0)) = \frac{\partial^2 \rho_\epsilon}{\partial w_i \partial \bar{w}_j}(z_0).$$

It follows from the considerations above that $\tilde{\rho}_\epsilon$ is plurisubharmonic on each stratum $S \subset X//G$. Since $\tilde{\rho}_\epsilon$ extends continuously to $X//G$, the results of [GR56] imply that $\tilde{\rho}_\epsilon$ is plurisubharmonic on $X//G$. Hence, $\tilde{\rho}$ is strictly plurisubharmonic.

Covering $X//G$ with sets $U_\alpha//G$, the corresponding strictly plurisubharmonic functions $\tilde{\rho}_\alpha$ fit together to define a Kähler structure on $X//G$ with the desired smoothness properties. We emphasize at this point that the induced Kähler structure does not depend on the choice of the pairs (U_α, ρ_α) .

Remark 1.11. Inspecting the proof that we have outlined above, one sees that Kählerian reduction works under the following weaker regularity assumptions: as before, let K be a compact Lie group and $G = K^\mathbb{C}$ its complexification. Let X be a holomorphic Kähler G -space with analytic Hilbert quotient $\pi_G : X \rightarrow X//G$. Let \mathbb{S} be a stratification of $X//G$ which is finer than the stratification of $X//G$ as a complex space and finer than the decomposition of $X//G$ by G -orbit types. Assume that the Kähler structure of X is smooth along every G -orbit in X , on a Zariski-open smooth subset X_{reg} of X and along the set of closed orbits in $\pi_G^{-1}(S)$ for each stratum S of $X//G$. Assume that the Kähler structure is K -invariant and that there exists a continuous map $\mu : X \rightarrow \mathfrak{k}^*$ which is a smooth momentum map for the K -action on X_{reg} as well as on every G -orbit. In this situation, we call X a *stratified Hamiltonian Kähler $K^\mathbb{C}$ -space*. Assume that $X = X(\mathcal{M})$. Then, the fundamental diagram (1) holds (in particular, $X//G$ is homeomorphic to $\mu^{-1}(0)/K$) and the construction outlined above yields a Kähler structure on $X//G$ which is smooth along the stratification \mathbb{S} .

The following example shows that even if the quotient is a smooth manifold, we cannot expect the reduced Kähler structure to be smooth. This illustrates that the reduction procedure is also sensible to singularities of the map $\mathcal{M} \rightarrow \mathcal{M}/K$.

Example 1.12. Consider $X = \mathbb{C}^2$ with the action of $\mathbb{C}^* = (S^1)^\mathbb{C}$ given by $t \bullet (z, w) = (tz, t^{-1}w)$. The standard Kähler form on \mathbb{C}^2 can be written as $\frac{i}{2}\partial\bar{\partial}\rho$, where $\rho : v \mapsto \|v\|^2$ denotes the square of the norm function associated to the standard Hermitean product on \mathbb{C}^2 . After identification of $Lie(S^1)^*$ with \mathbb{R} , the momentum map associated to ρ is given by $\mu(z, w) = |z|^2 - |w|^2$. It follows that the momentum zero fibre is singular at the origin. Every point in \mathbb{C}^2 is semistable and the analytic Hilbert quotient is realised by $\pi : \mathbb{C}^2 \rightarrow \mathbb{C}, (z, w) \mapsto zw$. Restriction of ρ to $\mu^{-1}(0)$ induces the function $\tilde{\rho}(\tilde{z}) = |\tilde{z}|$ on $\mathbb{C}^2//\mathbb{C}^* = \mathbb{C}$. It is continuous strictly plurisubharmonic and smooth along the orbit type stratification of \mathbb{C} .

2 Reduction in steps

From now on we consider the following situation: Let K be a connected compact Lie group, $G = K^{\mathbb{C}}$ its complexification and let X be a connected Hamiltonian Kähler $K^{\mathbb{C}}$ -manifold with K -equivariant momentum map $\mu : X \rightarrow \mathfrak{k}^*$. Let L be a closed normal subgroup of K . Then, $L^{\mathbb{C}} =: H$ is contained in $K^{\mathbb{C}}$ as a closed normal complex subgroup. The inclusion $\iota : \mathfrak{l} \rightarrow \mathfrak{k}$ of $\mathfrak{l} = \text{Lie}(L)$ into \mathfrak{k} induces an adjoint map $\iota^* : \mathfrak{k}^* \rightarrow \mathfrak{l}^*$. The composition $\mu_L : X \rightarrow \mathfrak{l}^*$, $\mu_L := \iota^* \circ \mu$ is a momentum map for the action of L on X . Set $\mathcal{M}_L := \mu_L^{-1}(0)$. There is a corresponding set of semistable points $X(\mathcal{M}_L) := \{x \in X \mid \overline{H \cdot x} \cap \mathcal{M}_L \neq \emptyset\}$, and an analytic Hilbert quotient $\pi_H : X(\mathcal{M}_L) \rightarrow X(\mathcal{M}_L)//H$. In the following we will investigate the relations between this quotient and the quotient $\pi_G : X(\mathcal{M}) \rightarrow X(\mathcal{M})//G$. The main result we show here is

Theorem 2.1 (Kählerian reduction in steps). *With the notation introduced above, the following holds:*

- a) *The analytic Hilbert quotient $Q_L := X(\mathcal{M})//H$ exists and is realised as an open subset of $X(\mathcal{M}_L)//H$. There is a holomorphic G -action on Q_L such that the quotient map $\pi_H : X(\mathcal{M}) \rightarrow Q_L$ is G -equivariant. The analytic Hilbert quotient $\bar{\pi} : Q_L \rightarrow Q_L//G$ exists. It is naturally biholomorphic to $X(\mathcal{M})//G$ and the following diagram commutes*

$$\begin{array}{ccccc} X(\mathcal{M}_L) & \hookleftarrow & X(\mathcal{M}) & \xrightarrow{\pi_G} & X(\mathcal{M})//G \\ \pi_H \downarrow & & \pi_H \downarrow & \nearrow \bar{\pi} & \\ X(\mathcal{M}_L)//H & \hookleftarrow & Q_L & & \end{array} .$$

- b) *The restriction of the momentum map μ to \mathcal{M}_L is L -invariant and induces a momentum map for the K -action on Q_L . This makes Q_L into a stratified Hamiltonian Kähler $K^{\mathbb{C}}$ -space. The analytic Hilbert quotient $Q_L//G$ carries a Kählerian structure induced by the Kählerian structure of Q_L . This Kählerian structure coincides with the Kählerian structure obtained by reduction for the quotient $\pi_K : \mathcal{M} \rightarrow \mathcal{M}/K \simeq X(\mathcal{M})//G$.*
- c) *The G -orbit type stratification of $X(\mathcal{M})//G$ coincides with the 2-step stratification which is defined in Section 2.2.*

2.1 Analytic reduction in steps

In this section we will prove part a) of Theorem 2.1.

Theorem 2.2 (Analytic reduction in steps). *Let X be a holomorphic G -space for the complex-reductive Lie group G and let $H \triangleleft G$ be a reductive normal subgroup. Assume that the analytic Hilbert quotient $\pi_G : X \rightarrow X//G$ exists. Then, the analytic Hilbert quotient $\pi_H : X \rightarrow X//H$ exists and admits a holomorphic G -action such that π_H is G -equivariant. Furthermore, the analytic Hilbert quotient of $X//H$ by G exists and is naturally biholomorphic to $X//G$. If $\bar{\pi} : X//H \rightarrow X//G$*

denotes the quotient map, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi_G} & X//G \\ \pi_H \downarrow & \nearrow \bar{\pi} & \\ X//H & & \end{array} \quad (3)$$

commutes.

Proof. The quotient map $\pi_G : X \rightarrow X//G$ is an H -invariant Stein map. Hence, the quotient $X//H$ exists (see [HMP98]). Let $\pi_H : X \rightarrow X//H$ be the quotient map. The map

$$\text{id}_G \times \pi_H : G \times X \rightarrow G \times X//H$$

is an analytic Hilbert quotient for the H -action on $G \times X$ which is given by the action of H on the second factor. Since H is a normal subgroup of G , the map that is obtained by composition of the action map $G \times X \rightarrow X$ with the quotient map π_H is an H -invariant holomorphic map from $G \times X$ to $X//H$. By the universal property of analytic Hilbert quotients we obtain a holomorphic map $G \times X//H \rightarrow X//H$ such that the following diagram commutes:

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ \text{id}_G \times \pi_H \downarrow & & \downarrow \pi_H \\ G \times X//H & \longrightarrow & X//H. \end{array}$$

This defines a holomorphic action of G on $X//H$ such that π_H is G -equivariant.

The G -invariant map $\pi_G : X \rightarrow X//G$ descends to a G -invariant map $\bar{\pi} : X//H \rightarrow X//G$. We claim that $\bar{\pi}$ is Stein. Indeed, let $U \subset X//G$ be Stein. Since π_G is an analytic Hilbert quotient map, the inverse image $\pi_G^{-1}(U)$ is a π_H -saturated Stein subset of X . Since analytic Hilbert quotients of Stein spaces are Stein (see [Hei91]), $\pi_H(\pi_G^{-1}(U)) = \bar{\pi}^{-1}(U)$ is a Stein open set in $X//H$. Furthermore, using the fact that $\bar{\pi}$ is induced by π_G and that π_H is G -equivariant, we see that $\mathcal{O}_{X//G} = \bar{\pi}_*(\mathcal{O}_{X//H})^G$. This shows that the map $\bar{\pi} : X//H \rightarrow X//G$ is the analytic Hilbert quotient of $X//H$ by the action of G . \square

In the situation of Theorem 2.1, it follows from the previous theorem that the analytic Hilbert quotient of $X(\mathcal{M})$ by the action of a normal complex-reductive subgroup H of G exists. For our purposes, it is important to relate this quotient to the quotient of $X(\mathcal{M}_L)$ by H and to the momentum geometry of μ_L . For later reference, we make the following

Definition 2.3. Let X be a holomorphic G -space and $A \subset X$. We define $\mathcal{S}_G(A) := \{x \in X \mid \overline{G \cdot x} \cap A \neq \emptyset\}$ and call it the *saturation* of A with respect to G . If the space X plays a role in our considerations, we also write $\mathcal{S}_G^X(A)$.

Lemma 2.4. Let L be a compact subgroup of K and $H := L^\mathbb{C} \subset G = K^\mathbb{C}$. Then, the set $X(\mathcal{M})$ is a π_H -saturated subset of $X(\mathcal{M}_L)$.

Proof. Let $x_0 \in X(\mathcal{M})$. By Proposition 1.9 there exists a strictly plurisubharmonic exhaustion function ρ of the fibre $F := \pi_G^{-1}(\pi_G(x_0))$ with the property that $\mu|_F$ is associated to ρ . This implies that $\mathcal{M} \cap F$ is the set where $\rho|_F$ assumes its minimum. The restriction of ρ to $C := \overline{H \bullet x_0} \cap X(\mathcal{M}) \subset F$ also is an exhaustion. This implies that $\rho|_C$ attains its minimum at some point $y_0 \in \overline{H \bullet x_0}$. For all $\xi \in \mathfrak{l}$, we have $\mu_L^\xi(y_0) = \frac{d}{dt}|_{t=0} \rho(\exp(it\xi) \bullet y_0) = 0$. Hence, $y_0 \in \mathcal{M}_L$ and $x_0 \in X(\mathcal{M}_L)$.

The first part of the proof shows that $X(\mathcal{M}) \subset \mathcal{S}_H(X(\mathcal{M}) \cap \mathcal{M}_L)$. Conversely, let $x \in \mathcal{S}_H(X(\mathcal{M}) \cap \mathcal{M}_L)$. Then, by definition, $\overline{H \bullet x} \cap (X(\mathcal{M}) \cap \mathcal{M}_L) \neq \emptyset$. Since $X(\mathcal{M})$ is an open H -invariant neighbourhood of $X(\mathcal{M}) \cap \mathcal{M}_L$, this implies $x \in X(\mathcal{M})$. Hence, we have shown that $X(\mathcal{M}) = \mathcal{S}_H(X(\mathcal{M}) \cap \mathcal{M}_L)$. This concludes the proof. \square

As before, let $\pi_G : X(\mathcal{M}) \rightarrow X(\mathcal{M})//G$ and $\pi_H : X(\mathcal{M}_L) \rightarrow X(\mathcal{M}_L)//H$ denote the quotient maps. From Lemma 2.4, we obtain

Corollary 2.5. *The analytic Hilbert quotient for the H -action on $X(\mathcal{M})$ is given by $\pi_H|_{X(\mathcal{M})} : X(\mathcal{M}) \rightarrow \pi_H(X(\mathcal{M})) \subset X(\mathcal{M}_L)//H$. In particular, $X(\mathcal{M})//H$ can be realised as an open subset of $X(\mathcal{M}_L)//H$.*

Theorem 2.2 and Corollary 2.5 prove part a) of Theorem 2.1.

As a preparation for the proof of part b) of Theorem 2.1, we will now relate the preceding discussion to the actions of the groups K and L .

Lemma 2.6. *The momentum zero fibre \mathcal{M}_L of μ_L as well as the set of semistable points $X(\mathcal{M}_L)$ is K -invariant.*

Proof. Let $x \in \mathcal{M}_L$. Since L is normal in K , we have $\text{Ad}(K)(\mathfrak{l}) \subset \mathfrak{l}$. This implies $\mu^\xi(k \bullet x) = \text{Ad}^*(k)(\mu(x)) = 0$. Hence, we have $k \bullet x \in \mathcal{M}_L$.

Since \mathcal{M}_L is K -invariant and $hH = Hk$ holds for all $k \in K$, we have $\overline{H \bullet (k \bullet x)} \cap \mathcal{M}_L = k \bullet (\overline{H \bullet x} \cap \mathcal{M}_L)$ for all $x \in X(\mathcal{M}_L)$. This shows the claim. \square

Since $X(\mathcal{M}_L)$ is K -invariant, it follows by considerations analogous to those in the proof of Theorem 2.2 that $X(\mathcal{M}_L)//H$ is a complex K -space. On $\pi_H(X(\mathcal{M}))$ the K -action coincides with the restriction of the $K^\mathbb{C}$ -action to K . However, we do not know if the action defined in this way on $\pi_H(X(\mathcal{M})) \subset X(\mathcal{M}_L)//H$ extends to a holomorphic action of $K^\mathbb{C}$ on $X(\mathcal{M}_L)//H$ in general.

In two important special cases, there is a holomorphic $K^\mathbb{C}$ -action on $X(\mathcal{M}_L)//H$. If $X(\mathcal{M}_L)//H$ is compact, its group of holomorphic automorphisms \mathcal{A} is a complex Lie group. The action of K on $X(\mathcal{M}_L)//H$ yields a homomorphism of K into \mathcal{A} . This extends to a holomorphic homomorphism of $K^\mathbb{C}$ into \mathcal{A} by the universal property of $K^\mathbb{C}$, hence $K^\mathbb{C}$ acts holomorphically on $X(\mathcal{M}_L)//H$. The second case is the following: as we have seen in Lemma 2.6, the complement of $X(\mathcal{M}_L)$ in X is K -invariant. If it is an analytic subset of X , its K -invariance implies its $K^\mathbb{C}$ -invariance. In this case, $K^\mathbb{C}$ acts on $X(\mathcal{M}_L)$ and hence on $X(\mathcal{M}_L)//H$.

Lemma 2.7. *We have $\mathcal{S}_G^{\pi_H(X(\mathcal{M}))}(\pi_H(\mathcal{M})) = \pi_H(X(\mathcal{M}))$.*

Proof. Let $x \in X(\mathcal{M})$. Then, by definition, $\overline{G \bullet x} \cap \mathcal{M} \neq \emptyset$. This implies that $\overline{G \bullet \pi_H(x)} \supset \pi_H(\overline{G \bullet x})$ intersects $\pi_H(\mathcal{M})$ non-trivially. This shows the claim. \square

2.2 Kähler reduction in steps

In this section we will prove part b) of Theorem 2.1. The results of Section 2.1 show that it is sensible to restrict to the situation where $X = X(\mathcal{M}) = X(\mathcal{M}_L)$ for the discussion of Kählerian reduction in this section and we will do this from now on.

First we take a closer look at the compatibility of the G -action on $X//H = X(\mathcal{M})//H$ with the orbit type stratification $\mathbb{S}_H^{X//H}$ of $X//H$. For later reference, we note the following

Lemma 2.8. *Let M be a Lie group with finitely many connected components and M_0 a closed subgroup of M . Then, the following are equivalent:*

1. M and M_0 are isomorphic as topological groups,
2. $M_0 = M$.

As a first application, we get

Lemma 2.9. *The stratification $\mathbb{S}_H^{X//H}$ is G -invariant.*

Proof. Let $z \in \mathcal{M}_L$ and consider the H -action on $G \bullet z$. The orbit $H \bullet z$ is closed. Since $H \bullet g \bullet z = g \bullet H \bullet z$ holds for all $g \in G$, all H -orbits in $G \bullet z$ are closed (and have the same dimension). Since $G \bullet z$ is connected, there is a principal H -stratum S in $G \bullet z$ and we may assume $z \in S$. For any $g \in G$ this yields $hH_z h^{-1} \subset H_g \bullet z = g(H \cap G_z)g^{-1} = gH_z g^{-1}$ for some $h \in H$. Lemma 2.8 implies $hH_z h^{-1} = gH_z g^{-1} = H_g \bullet z$ and therefore $G \bullet z = S$. \square

We now investigate the compatibility of the $K^\mathbb{C}$ -action on $X//H$ with the induced Kähler structure.

Proposition 2.10. 1. *The reduced Kähler structure of $X//H$ is smooth along each G -orbit in $X//H$.*

2. *The reduced Kähler structure is K -invariant. The restriction of μ to \mathcal{M}_L is L -invariant and induces a well-defined continuous map $\tilde{\mu} : X//H \rightarrow \mathfrak{k}^*$ which is a smooth momentum map on each stratum of $X//H$ as well as on every G -orbit.*

Proof. The induced Kähler structure $\tilde{\omega}$ is smooth along the stratification $\mathbb{S}_H^{X//H}$ by construction. Furthermore, Lemma 2.9 shows that this stratification is G -invariant. Hence, $\tilde{\omega}$ is smooth along every G -orbit in $X//H$.

Let $x \in \mathcal{M}_L \subset X$. Applying Proposition 1.9 to X , to the momentum map $\mu : X \rightarrow \mathfrak{k}^*$ and to the quotient $\pi_G : X \rightarrow X//G$, we get a π_G -saturated neighbourhood U of x in X on which the Kähler structure and the momentum maps μ and μ_L are induced by a K -invariant strictly plurisubharmonic function $\rho : U \rightarrow \mathbb{R}$. Since the Kähler structure on $\pi_H(U) \subset X//H$ is induced by the restriction of ρ to $\mathcal{M}_L \cap U$ it is K -invariant.

For the L -invariance of $\mu|_{\mathcal{M}_L}$ we follow [SL91]: first, we equivariantly identify \mathfrak{k} with \mathfrak{k}^* . The image of $\mu : \mathcal{M}_L \rightarrow \mathfrak{k}$ is contained in $\mathfrak{l}^\perp \subset \mathfrak{k}$. Here, $^\perp$ denotes the perpendicular with respect to a chosen K -invariant inner product on \mathfrak{k} . Hence, the equation $\mu(l \cdot x) = \text{Ad}(l)(\mu(x))$ implies that it is sufficient to show that L acts trivially on \mathfrak{l}^\perp . As \mathfrak{l}^\perp is L -invariant, $[\mathfrak{l}, \mathfrak{l}^\perp]$ is contained in \mathfrak{l}^\perp . But \mathfrak{l} is an ideal in \mathfrak{k} and hence, $[\mathfrak{l}, \mathfrak{l}^\perp]$ is also contained in \mathfrak{l} . This implies that the component of the identity L_0 of L acts trivially on \mathfrak{l}^\perp . Since K is connected, the image of the map $\phi : K \times L_0 \rightarrow K, (k, l) \mapsto k l k^{-1}$ lies in some connected component of L , which has to be L_0 . Hence, L_0 is normal in K . The finite group L/L_0 is normal in the connected group K/L_0 , hence central. It follows that L/L_0 acts trivially on $\text{Lie}(K/L_0) = \mathfrak{l}^\perp$. This shows that L acts trivially on \mathfrak{l}^\perp .

Hence, $\mu|_{\mathcal{M}_L}$ defines a continuous map $\tilde{\mu}$ on $X//H$. This map is a smooth momentum map along each stratum S of $X//H$, since $\pi_L : \mathcal{M}_L \cap \pi_H^{-1}(S) \rightarrow \tilde{S}$ is a smooth K -equivariant submersion. Since the stratification is G -invariant, $\tilde{\mu}$ is a smooth momentum map along each G -orbit in $X//H$. \square

Remark 2.11. By construction, we see that $\tilde{\mathcal{M}} := \tilde{\mu}^{-1}(0)$ is equal to $\pi_H(\mathcal{M})$. Furthermore, Lemma 2.7 shows that $\mathcal{S}_G(\tilde{\mathcal{M}}) = X//H$.

We will now investigate the possibility of carrying out Kähler reduction with respect to the quotient $\bar{\pi} : X//H \rightarrow X//G$.

First we define a second stratification of $X//G$ as follows: let S be a stratum of the H -orbit type stratification of $X//H$ such that $S \cap \tilde{\mathcal{M}} \neq \emptyset$. Stratify $\bar{\pi}(\bar{S}) \setminus \bar{\pi}(\partial S)$ by G -orbit types with respect to the G -action on $X//H$. Repeating this procedure for all strata yields a stratification of $X//G$ which we call the *2-step-stratification*. We will see later on that it coincides with the stratification of $X//G$ by G -orbit types (see Section 2.3).

We notice that by Proposition 2.10 and by the construction of the 2-step stratification $X//H$ is a stratified Hamiltonian Kähler $K^\mathbb{C}$ -space. We also note that by the fundamental diagram (1), $X//G$ is homeomorphic to $\tilde{\mathcal{M}}/K$. Applying the procedure described in Section 1.3, we have

Proposition 2.12. *The analytic Hilbert quotient $X//G$ carries a Kähler structure induced from $X//H$ by Kählerian reduction for the quotient $\tilde{\pi}_K : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}}/K$. This Kähler structure is smooth along the 2-step stratification.*

We now have two Kählerian structures on the complex space $X//G$: the structure ω_{red} that we get by reducing the Kähler form ω on X to $X//G$ and the Kähler structure $\hat{\omega}$ that we get by first reducing ω to the Kähler structure $\tilde{\omega}$ on $X//H$ and then reducing $\tilde{\omega}$ to $\hat{\omega}$ on $X//G$. In order to complete the proof of Theorem 2.1 b), we have to prove that ω_{red} and $\hat{\omega}$ coincide.

Let $y_0 \in X//G$. Proposition 1.9 yields a open neighbourhood U of y_0 in $X//G$ such that on $\pi_G^{-1}(U)$ the Kähler structure and the momentum maps μ and μ_L are given by ρ . It follows that the Kähler structure on $\bar{\pi}^{-1}(U)$ is given by $\tilde{\rho}$ which is induced via $\rho|_{\mathcal{M}_L \cap \pi_G^{-1}(U)}$.

Lemma 2.13. *$\tilde{\rho}$ is an exhaustion along every fibre of $\bar{\pi}$. On $\bar{\pi}^{-1}(U)$, we have $\tilde{\mu} = \mu_{\tilde{\rho}}$.*

Proof. Let $y \in U$ and let $F := \pi_G^{-1}(y)$. We have $\mathcal{M} \cap F = K \cdot x$ for some $x \in F$. The restriction $\rho|_F$ is an exhaustion. It is minimal along $\mathcal{M} \cap F = K \cdot x$. Since $\tilde{F} := \bar{\pi}^{-1}(y) = \pi_H(F)$ and since $\tilde{\rho}|_{\tilde{F}}$ is induced via the restriction of ρ to $\mathcal{M}_L \cap F$, $\tilde{\rho}|_{\tilde{F}}$ is minimal along $\tilde{\mathcal{M}} \cap \tilde{F} = K \cdot \pi_H(x)$. It follows that $\tilde{\rho}|_{\tilde{F}}$ is an exhaustion.

Since both $\tilde{\mu}$ and $\mu_{\tilde{\rho}}$ are momentum maps for the K -action on $\bar{\pi}^{-1}(U)$, they differ by a constant $c \in \mathfrak{k}^*$. Hence, it suffices to show that there exists an $z \in \bar{\pi}^{-1}(U)$ such that $\tilde{\mu}(z) = \mu_{\tilde{\rho}}(z)$ holds. Let $z_0 \in \tilde{\mathcal{M}} \cap \bar{\pi}^{-1}(U)$. As we have seen above, $\tilde{\rho}|_{\bar{\pi}^{-1}(\bar{\pi}(z_0))}$ is critical along $K \cdot z_0$. Therefore, $\mu_{\tilde{\rho}}(z_0) = \tilde{\mu}(z_0) = 0$ and hence $c = 0$. \square

The equality of the two Kähler structures on U now follows from the construction of reduced structures: the Kähler structure ω_{red} is formed by a single function ρ_{red} which is induced by the restriction of ρ to $\mathcal{M} \cap \pi_G^{-1}(U)$. The Kähler structure $\tilde{\omega}$ on $\bar{\pi}^{-1}(U)$ is given by $\tilde{\rho}$. Lemma 2.13 shows that the Kähler structure $\hat{\omega}$ is given by a single function $\hat{\rho}$ that is induced by the restriction of $\tilde{\rho}$ to $\tilde{\mathcal{M}} \cap \bar{\pi}^{-1}(U)$. However, $\tilde{\mathcal{M}} = \mathcal{M}/L$ and therefore, $\hat{\rho}$ coincides with ρ_{red} after applying the homeomorphism $(\mathcal{M}/L)/K \simeq \mathcal{M}/K$. This concludes the proof of Theorem 2.1.

2.3 Stratifications in steps

Here we prove part c) of Theorem 2.1. We formulate it as

Proposition 2.14. *The stratification of $X//G$ by G -orbit types coincides with the 2-step-stratification of $X//G$.*

Proof. We do this in two steps: first, we claim that the G -stratification of $X//G$ is finer than the 2-step-stratification. Let S be a stratum of the G -stratification of $X//G$ and let Y be the set of closed orbits in $\pi_G^{-1}(S)$. Then, since S and G are connected, Y is a connected holomorphic H -manifold.

Every H -orbit in Y is closed. Stratify Y by H -orbit types. Since Y is connected, there exists a principal orbit type (H_0) for the H -action on Y . Then, $H_0 = G_0 \cap H$, where (G_0) is the orbit type of the G -stratum S . Let $y \in Y$. We have $H_y = gH_0g^{-1}$ for some $g \in G$. Since (H_0) is the principal isotropy type, there exists an $h \in H$ such that $hH_0h^{-1} < H_y$. Lemma 2.8 implies that H_y is conjugate to H_0 in H . Hence, $\pi_H(Y)$ is a union of closed G -orbits that is contained in a single stratum $\mathbb{S}_H^{X//H}$. Furthermore, the G -isotropy group of each element in $\pi_H(Y)$ is conjugate in G to G_0H . This implies that the G -stratification of $X//G$ is finer than the 2-step-stratification.

In the second step, we prove that the 2-step-stratification of $X//G$ is finer than the G -stratification of $X//G$. Let S be a stratum of the 2-step stratification of $X//G$. Let $w, w' \in \pi_G^{-1}(S) \cap \mathcal{M}$. Then, by construction of S , we have

1. H_w is conjugate to $H_{w'}$ in H and
2. HG_w is conjugate to $HG_{w'}$ in G .

We claim that $\dim G_w = \dim G_{w'}$. Indeed, 1) and 2) imply the following dimension equalities on the Lie algebra level:

$$\begin{aligned} \dim \mathfrak{g}_w &= \dim(\mathfrak{g}_w + \mathfrak{h}) - \dim \mathfrak{h} + \dim(\mathfrak{g}_w \cap \mathfrak{h}) \\ &= \dim(\mathfrak{g}_{w'} + \mathfrak{h}) - \dim \mathfrak{h} + \dim(\mathfrak{g}_{w'} \cap \mathfrak{h}) \\ &= \dim \mathfrak{g}_{w'} \end{aligned}$$

Hence, $\dim G \bullet w = \dim G \bullet w' =: m_0$. The set

$$Y := \{x \in \pi_G^{-1}(S) \mid G \bullet x \text{ closed in } X\} = \pi_G^{-1}(S) \cap \{x \in X \mid \dim G \bullet x \leq m_0\}$$

is analytic in $\pi_G^{-1}(S)$. There exists a principal orbit type (G_0) for the action of G on Y . Let $w \in Y$ such that $(G_w) = (G_0)$. Let w' be any other point in Y . Since (G_w) is the principal orbit type for the action on Y , there exists a $g_0 \in G$ such that $g_0 G_w g_0^{-1} \subset G_{w'}$. Hence, we can assume that $G_w \subset G_{w'}$. Let $\iota : G_w \rightarrow G_{w'}$ be the inclusion. Since $G_w \subset G_{w'}$, we also have $H_w \subset H_{w'}$. Together with assumption 1.) and Lemma 2.8, this implies that $H_w = H_{w'}$. If we let $H_w = H_{w'}$ act on G_w and $G_{w'}$ as a normal subgroup, ι is clearly equivariant. This shows that ι induces an injective group homomorphism $\tilde{\iota} : G_w/H_w \rightarrow G_{w'}/H_w$. For every $y \in Y$ we have the following exact sequence

$$1 \rightarrow H_y \rightarrow G_y \rightarrow G_y H/H \rightarrow 1.$$

Together with assumption 2.), this implies that G_w/H_w and $G_{w'}/H_w$ are isomorphic as topological groups. Since both G_w/H_w and $G_{w'}/H_w$ have a finite number of connected components, Lemma 2.8 implies that $\tilde{\iota}$ is surjective. It follows that ι is surjective and that $G_{w'}$ is equal to G_w . This shows that the 2-step stratification is finer than the G -stratification of $X//G$ and we are done. \square

3 Projectivity and Kählerian reduction

One source of examples of quotients for actions of complex-reductive Lie groups is Geometric Invariant Theory (GIT). In this section we will explain the construction of GIT-quotients in detail, noting that it is an example of the Kählerian reduction in steps procedure yielding projective algebraic quotient spaces. In addition we will discuss projectivity results for general Hamiltonian actions on complex algebraic varieties and their compatibility with reduction in steps.

A complex-reductive group $G = K^{\mathbb{C}}$ carries a uniquely determined algebraic structure making it into a linear algebraic group. In this section we study algebraic actions of complex-reductive

groups G with respect to this algebraic structure. For this, we define a *complex algebraic G -variety* X to be a complex algebraic variety X together with an action of G such that the action map $G \times X \rightarrow X$ is algebraic.

Let X be a projective complex algebraic G -variety. Let L be a very ample G -linearised line bundle on X , i.e. a very ample line bundle L on X and an action of G on L by bundle automorphisms making the bundle projection G -equivariant. The action of G on X and on L induces a natural representation on $V := \Gamma(X, L)^*$ and there exists a G -equivariant embedding of X into $\mathbb{P}(V)$. Let

$$\mathcal{N}(V) = \{v \in V \mid f(v) = 0 \text{ for all } f \in \mathbb{C}[V]^G\}$$

be the *nullcone* of V . Here $\mathbb{C}[V]^G$ denotes the algebra of polynomials on V that are invariant under the action of G . Let $X(V) := X \setminus p(\mathcal{N}(V))$, where $p : V \setminus \{0\} \rightarrow \mathbb{P}(V)$ denotes the projection. Then, it is proven in Geometric Invariant Theory that the analytic Hilbert quotient $\pi_G : X(V) \rightarrow X(V)//G$ exists, that $X(V)//G$ is a projective algebraic variety, that the quotient map is algebraic and affine, and that the algebraic structure sheaf on the quotient is the sheaf of invariant polynomials on $X(V)$ (see [MFK94]). In this situation, we call π an *algebraic Hilbert quotient*. We will explain the relation of this algebraic quotient theory to reduction in steps. Consider the action of the complex-reductive group $G \times \mathbb{C}^*$ on V that is given by the G -representation and the action of \mathbb{C}^* by multiplication. By a theorem of Hilbert, the analytic Hilbert quotient $V//G$ exists as an affine algebraic variety. It can be embedded into a \mathbb{C}^* -representation space W with $\mathbb{C}[W]^{\mathbb{C}^*} = \mathbb{C}$ as a \mathbb{C}^* -invariant algebraic subvariety. The orbit space $W \setminus \{0\}/\mathbb{C}^*$ is a weighted projective space. It follows that the analytic Hilbert quotient of $V \setminus \mathcal{N}(V)$ by the action of $G \times \mathbb{C}^*$ exists as a projective algebraic variety. Let $C(X) := \overline{\pi^{-1}(X)} \subset V$ be the *cone over X* . The considerations above imply that the analytic Hilbert quotient of $C(X) \setminus \mathcal{N}(V)$ by the action of $G \times \mathbb{C}^*$ exists as a projective algebraic variety. Now we note that $X(V) = (C(X) \setminus \mathcal{N}(V))/\mathbb{C}^*$, i.e. $X(V)$ is the quotient of $C(X) \setminus \mathcal{N}(V)$ by the normal subgroup \mathbb{C}^* of $G \times \mathbb{C}^*$. By Theorem 2.2, the analytic Hilbert quotient $\pi_G : X(V) \rightarrow X(V)//G$ exists and the following diagram commutes:

$$\begin{array}{ccc} C(X) \setminus \mathcal{N}(V) & \longrightarrow & X(V)//G \\ \downarrow p & \nearrow \pi_G & \\ X(V) & & \end{array} .$$

The variety $C(X)$ is Kähler with Kähler structure given by the square of the norm function associated to a K -invariant Hermitean product $\langle \cdot, \cdot \rangle$ on V . Furthermore, the action of $K \times S^1$ is Hamiltonian. A momentum map is given by

$$\mu^{(\xi, \sqrt{-1})}(v) = 2i \langle \xi v, v \rangle + \|v\|^2 - 1 \quad \forall \xi \in \mathfrak{k}.$$

Here, $\sqrt{-1}$ is considered as an element of $\text{Lie}(S^1)$. The set of semistable points coincides with $C(X) \setminus \mathcal{N}(V)$. The set of semistable points for the \mathbb{C}^* -action and the restricted momentum map is $V \setminus \{0\}$. The Fubini-Study-metric on X is obtained by Kählerian reduction for the quotient $p : C(X) \setminus \{0\} \rightarrow X$ and the K -action on X is Hamiltonian with respect to the Fubini-Study-metric with momentum map $\tilde{\mu}^\xi([v]) = \frac{2i \langle \xi v, v \rangle}{\|v\|^2}$. The set of semistable points with respect to $\tilde{\mu}$ coincides with $X(V)$.

We will now take a closer look at the Kählerian structure that we get by Kählerian reduction of the Fubini-Study-metric to $X(V)//G \simeq \tilde{\mu}^{-1}(0)/K$. We have already noted that $X(V)//G$ is projective algebraic. It carries an ample line bundle L_{red} such that there exists an $n_0 \in \mathbb{N}$ with $\pi_G^*(L_{red}) = H^{n_0}$, where H denotes the restriction of the hyperplane bundle to $X(V)$. With this notation, the reduced Kähler structure ω_{red} on $X(V)//G$ fulfills

$$n_0 \cdot c_1(\omega_{red}) = c_1(L_{red}) \in H^2(X, \mathbb{R}),$$

where c_1 denotes the first Chern class of ω_{red} and of L_{red} , respectively. Hence, the cohomology class of the Kähler structure obtained by Kählerian reduction of the Fubini-Study-metric lies in the real span of the ample cone of $X(V)//G$.

The discussion above shows that analytic Hilbert quotients obtained by Kählerian reduction of projective algebraic varieties are again projective algebraic if the K -actions under consideration are Hamiltonian with respect to the Fubini-Study-metric. For arbitrary Kähler forms on an algebraic variety, i.e. forms that are not the curvature form of a very ample line bundle, and associated momentum maps there is no obvious relation to ample G -line bundles and the corresponding analytic Hilbert quotients of sets of semistable points are not a priori algebraic.

The best algebraicity result for momentum map quotients known so far is proven in [Gre]:

Let K be a compact Lie group and $G = K^{\mathbb{C}}$. Let X be a smooth complex algebraic G -variety such that the K -action on X is Hamiltonian with respect to a K -invariant Kähler form on X with momentum map $\mu : X \rightarrow \mathfrak{k}^$. Let $\mathcal{M} := \mu^{-1}(0)$, and let $\pi_G : X(\mathcal{M}) \rightarrow X(\mathcal{M})//G$ denote the quotient map. If \mathcal{M} is compact, the following holds:*

1. *The analytic Hilbert quotient $X(\mathcal{M})//G$ is (the complex space associated to) a projective algebraic variety.*
2. *There exists a G -equivariant biholomorphic map Φ from $X(\mathcal{M})$ to an algebraic G -variety Y , there exists an algebraic Hilbert quotient $p_G : Y \rightarrow Y//G$, and the map $\bar{\Phi} : X(\mathcal{M})//G \rightarrow Y//G$ that is induced by Φ is an isomorphism of algebraic varieties.*
3. *The algebraic G -variety Y is uniquely determined up to G -equivariant isomorphism of algebraic varieties.*

As before, let $L \triangleleft K$ be a normal closed subgroup of K and $H = L^{\mathbb{C}}$. Then, reduction in steps is compatible with the algebraicity results obtained above in the following way:

We have already seen that the analytic Hilbert quotient $\pi_H : X(\mathcal{M}) \rightarrow X(\mathcal{M})//H$ exists. Furthermore, there exists an algebraic Hilbert quotient $p_H : Y \rightarrow Y//H$ for the action of H on Y and the map Φ induces a G -equivariant biholomorphic map $\Phi_H : X(\mathcal{M})//H \rightarrow Y//H$. In addition, the algebraic Hilbert quotient $(Y//H)//G$ exists and it is biregular to $Y//G$. If $\bar{p} : Y//H \rightarrow Y//G$

denotes the quotient map, the following diagram commutes:

$$\begin{array}{ccccc}
& X(\mathcal{M})//H & & & \\
& \uparrow \pi_H & \searrow \bar{\pi} & & \\
X(\mathcal{M}) & \xrightarrow{\pi_G} & X(\mathcal{M})//G & & \\
\downarrow \Phi & & \downarrow \bar{\Phi} & & \\
Y & \xrightarrow{p_G} & Y//G & & \\
\downarrow p_H & \nearrow \bar{p} & & & \\
Y//H & & & & .
\end{array}$$

Φ_H (curved arrow from $X(\mathcal{M})$ to $Y//H$)

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